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CHARACTERIZATION OF HAMILTONIAN INPUT-OUTPUT SYSTEMS

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Abstract

In this paper we review recent results on characterization of Hamiltonian Input-Output systems, which have been obtained by the authors and B. Jakubczyk. We give three characterizations and develop their interrelationship.

1. Introduction

In this paper we describe results obtained by the authors and B. Jackubczyk concerning realization theory for Hamiltonian input-output systems. The particular aspect we stress here is the characterization of Hamiltonian input-output systems as a subclass of all input-output systems. For our current purposes we define an input-output system as one which has a finite dimensional state space representation in the following form

$$\Sigma \qquad \dot{x} = f(x,u)\ ,\ x\varepsilon M,\ u\varepsilon \Omega \subset R^m$$
$$y = h(x,u) \qquad y\varepsilon\ R^p$$

where M is an analytic manifold and for each $u\varepsilon\Omega$, $x\to f(x,u)$ is a complete analytic vector field on M. We assume also that the maps $(x,u)\to f(x,u)$ $(x,u)\to h(x,u)$ are analytic. Let Σ^L denote the subclass defined by equations

$$\Sigma^L \qquad \dot{x} = g_o(x) + \sum_{i=1}^{m} u_i g_i(x)\ ,\ x\varepsilon M,\ u\varepsilon\Omega\subset R^M$$
$$y_i = H_j(x)\ , \qquad j=1\dots p \qquad y\varepsilon\ R^p.$$

Before we define the subclass of Hamiltonian input-output system we discuss some important external (state independent) representations of input-output systems. The input-output map representation of a system consists of a causal mapping $F:U[o,\infty)\to Y[o,\infty)$ where $U[(a,b)]$ is the class of Ω-valued piecewise constant functions on $[(a,b)]$, and Y is the class of R^p-valued continuous functions on $[(a,b)]$. F has a realization by an input-output system Σ in case there exists an initial state $x_o\varepsilon M$ such that the input output map defined by the initialized system Σ coincides with F. There are two specific parameterizations of input-outputs maps which interest us here. If we evaluate F on particular elements of $U[o,\infty)$ for a system Σ we obtain

$$F\big((t_1,u_1)(t_2,u_2)\dots(t_{k-1},u_{k-1})u_k\big)$$
$$= h_{u_r} \circ Y_{t_{r-1}}^{u_{k-1}} \circ \dots \circ Y_{t_1}^{u_1}(x_o) \tag{1}$$

where $(t_1,u_1)(t_2,u_2)\dots(t_{k-1},u_{k-1})u_k$, $t_i\varepsilon R^+$ denotes a piecewise constant control on $[o, \sum_{i=1}^{k-1} t_i]$, and if $h_u(x) = h(x,u)$, $f_u(x) = f(x,u)$ then $(t,x)\to Y_t^u(x)$ denotes the flow of the (complete) vector field f_u. We now write

$$Q(u_1\dots u_r) = \frac{\partial}{\partial t_1}\ \dots\ \frac{\partial}{\partial t_{r-1}}\Big|_{t_i=0} F\ \big((t_1,u_1)(t_2,u_2)\dots$$
$$(t_{k-1},u_{k-1})u_k\big). \tag{2}$$

Clearly $Q(u_1\dots u_k) = \big(f_{u_1}(\dots(f_{u_{k-1}}(h_{u_k})\dots\big)(x_o)$. Formally we may express the input-output map F in the form $F = \sum_{\omega\varepsilon M} Q(\omega)\omega$, where M is the monoid constructed from the set of words on Ω, see Jakubczyk [13], for further details concerning realization theory based on such representations. Note also that in the particular case of systems Σ^L the expressions $Q(u_1..u_k)$ specialize to the coefficients in the Chen series or generating series parameterization of the input-output map, (see Fliess [6] for details). For this reason we do not explicity need such a parameterization here. The second parameterization of the input-output map we consider applies only to the systems Σ^L, and is known as the Volterra series, which we denote by

$$y(t) = W_o(t) + \int_o^t W_1(t,\sigma_1)u(\sigma_1))d\sigma_1 + \dots\dots$$
$$+ \int_o^t\int_o^{\sigma_1}\dots\int_o^{\sigma_{k-1}} W_k(t,\sigma_1,\dots\sigma_k)u(\sigma_1))d\sigma_1\dots d\sigma_k$$
$$+ \dots\dots\ ,\quad t\geq 0. \tag{3}$$

Here $W_k(t,\sigma_1,\dots\sigma_k)$ is a k-linear R^p valued mapping on $\Omega C\ R^m$. See [4] for detailed expressions for the kernels in terms of the system data.

The input-output behavior representation of a system Σ consists of the set of all pairs (u,y) where $u\varepsilon\ U(-\infty,\infty)$ and $y\varepsilon\ Y(-\infty,\infty)$, such that there exists an absolutely continuous M valued function x on $(-\infty,\infty)$, with the property that the mapping $t\to[u(t), y(t), x(t)]$ satisfies the equations Σ a.e. on $(-\infty,\infty)$. Denote this set by Σ_e, and denote by Σ_i the set of all 3-tuples (u,y,x) where $(u,y)\varepsilon\Sigma_e$ and x is a corresponding state trajectory. Denote by $\Sigma_i^+(o)(x_o)$, the set of all 3-tuples (u,y,x) where u,y,x are functions defined on $[0,\infty)$ coinciding with the restriction of functions $\bar{u}$, $\bar{y}$, $\bar{x}$ on $(-\infty,\infty)$ such that $(\bar{u}, \bar{y}, \bar{x})\ \varepsilon\ \Sigma_i$ and $\bar{x}(o) = x_o$. $\Sigma_e^+(o)(x_o)$ is defined as the projection of $\Sigma_i^+(o)(x_o)$ onto the set of pairs (u,y). Note that $\Sigma_i^+(o)(x_o)$ is in fact the same object as the input output map F defined by system Σ initialized at x_o.

For initialized, analytic systems satisfying a completeness assumption the existence and uniqueness of minimal realizations of input-output maps is well understood, Sussmann [23], Jakubczyk [11]. Even in the C^k and local cases many results are still valid [12], [6]. However the corresponding theory based on the input-output behavior representation of input output systems is still in its infancy. (See Fliess [7] concerning the differential equation representation). In any case we shall not concern ourselves here with conditions under which a given external representation admits a realization by an input-output system.

We say Σ is a Hamiltonian input-output system if M can be given the structure of symplectic manifold (M,ω), with sympletic form ω, with respect to which for each fixed $u\epsilon\Omega$, $f(x,u)$ is a locally Hamiltonian vector field, and for each $x_o \epsilon M$ there exists a neighborhood U of x_o such that on U, $\sum$ can be written in the form

$$\textstyle\sum_H \qquad \dot{x} = X_{H_u}(x) \ , \quad y = \frac{\partial H}{\partial u}(x,u)$$

$$x\epsilon\ U \subset M \quad u \epsilon \Omega \quad R^p,\ m = p.$$

That is on U, for each fixed u, X_{H_u} is a global Hamiltonian vector field with Hamiltonian function $x \to H_u(x) = H(x, u)$, $(\omega(X_{H_u}, Z) = - dH_u(z)$ for all vector fields Z on U) See Van der Schaft [20] for details concerning this definition. Note that we do not insist that $f(x,u)$ is a global Hamiltonion vector field on M. In case of an input-output system $\sum^L$, this definition reduces to the fact that we may rewrite the system in the form

$$\textstyle\sum_H^L \qquad \dot{x} = g_o(x) + \sum_{i=1}^{m} u_i X_{H_i}(x) \ , \ x\epsilon M,\ u\epsilon\Omega \subset R^p$$

$$y_i = H_i(x) \ , \ 1 \leq i \leq p \ , \ m = p.$$

Here X_{H_i} are global Hamiltonian vector fields on M with Hamiltonian functions H_i, and g_o is a locally Hamiltonian vector field with locally defined Hamiltonian function H_o. We say $\sum$ is a secondary Hamiltonian input-output system it has a representation as above but in $\sum_H$ we have $y = H(x,u)$. There is some reason to believe that the first definition of Hamiltonian input-output system given above is more natural from the systems theory point of view. In this paper we consider the characterizations of Hamiltonian input-output systems in terms of the external representations described above. In section (2) we will discuss an intermediate state space characterization, which plays a vital role in the work developed by the authors[5], and summarized in [19]. In section (3) we will discuss the characterization of Hamiltonian systems in terms of the input-output representation, principally the work of B. Jakubczyk [10], [9], and in section (4) give a discussion of the characterization in terms of the input-output behavior as obtained by the authors.

2. The Self-Adjointness Criterion

For simplicity we consider only the class of systems $\sum^L$. Take an arbitrary, but fixed, input function $u(t)$, $t \ \epsilon\ [0,T]$, such that the solution $x(t)$ of $\sum^L$ remains within one coordinate neighborhood of M. This also yields an output $y(t)$, $t \ \epsilon\ [o,T]$. Along this input-state-output trajectory (u,x,y) the variational system is given by

$$\dot{v}(t) = Dg_0(x(t))\, v(t) + \sum_{j=1}^{m} u_j(t)\, Dg_j(x(t))\, v(t) + \sum_{j=1}^{m} u_j^v(t)\, g_j(x(t)) \qquad (4)$$

$$y_j^v(t) = DH_j\ (x(t))\, v(t),\ j = 1,\ldots,p,\ v(0) = 0\ \epsilon\ R^k$$

where D denotes taking the Jacobian matrix. Furthermore $u^v = (u_1^v, \ldots, u_m^v)$ and $y^v = (y_1^v, \ldots, y_p^v)$ denote the inputs and outputs of the variational system, and are called the variational inputs and outputs. This nomenclature is explained as follows. Let $(u(t,\epsilon),\ x(t,\epsilon),\ y(t,\epsilon))$, $t\ \epsilon\ [0,T]$, be a one-parameter family of solutions of $\sum^L$ with $u(t,o) = u(t)$, $x(t,0) = x(t)$ and $y(t,0) = y(t)$, $t\ \epsilon\ [0,T]$, called a variation of (u,x,y) then we have

$$(\delta u(t),\ \delta x(t),\ \delta y(t)) = (u^v(t),\ v(t), y^v(t)),\ t\epsilon[o,T]$$

where $(\delta u(t),\ \delta x(t),\ \delta y(t))$

$$= (\tfrac{\partial u}{\partial \epsilon}(t,0),\ \tfrac{\partial x}{\partial \epsilon}(t,0),\ \tfrac{\partial y}{\partial \epsilon}(t,0))$$

is called the variational field along (u,x,y).

Along this same trajectory $(u(t),\ x(t),\ y(t))$, $t\ \epsilon\ [0,T]$, the adjoint system is defined as the dual linear time-varying system

$$-\dot{p}(t) = Dg_0^T(x\ (t))p(t) + \sum_{j=1}^{m} u_j\ Dg_j^T\ (x(t))\ p(t) + \sum_{j=1}^{p} u_j^a(t)\ DH_j^T(x(t)) \qquad (5)$$

$$y_j^a(t) = g_j^T\ (x(t))\ p(t)\ j = 1,\ldots,\ m,\ p(0) = 0\ \epsilon\ R^k$$

with inputs $u^a = (u_1^a,\ u_p^a)$ and outputs $y^a = (y_1^a, \ldots y_m^a)$. For any input functions $u^v(t)$ and $u^a(t)$ it follows from (4) and (5) that

$$\frac{d}{dt}\ p^T\ (t)\ v\ (t) = (y^a(t))^T\ u^v(t) - (y^v(t))^T u^a(t) \qquad (6)$$

Moreover, if a system with inputs u^a and outputs y^a satisfies (6) for any u^v and y^v then it is equal to the adjoint system [5]. Hence the adjoint system is uniquely determined by the variational system. The variational and adjoint systems are only defined locally along a trajectory $(u(t),\ x(t),\ (y(t))$, $t\ \epsilon\ [a,b]$, such that $x(t)$ remains within one coordinate neighborhood. However, global (and coordinate free) definitions can be given if we combine the original system together with all its variational or adjoint systems. Equations ($\sum^L$ together with (11) define the prolonged system, or prolongation, which has state space TM (local coordinates (x,v)), input space T R^m (local coordinates $(u,\ u^v)$) and output space T R^p (local coordinates $(y,\ y^v)$). Equations $\sum^L$ together with (5) define the Hamiltonian extension, which has state space T*M (local coordinates (x,p)), input space $R^m \times R^p$ (local coordinates (u,u^a)) and output space $R^p \times R^m$ (local coordinates (y,y^a)).

The input-output map of the variational system along a trajectory $(u(t),\ x(t),y(t))$ of $\sum^L$ is given by

$$y^v(t) = \int_0^t W_v\ (t,\sigma,u)\ u^v(\sigma)\ d\sigma,\ t,\sigma \geq 0 \qquad (7)$$

where $W_v(t,\sigma,u)$ is the pxm matrix with (i,j) - th element

$$DH_i\ (x(t))\ \Phi^u\ (t,\sigma)\ g_j\ (x(\sigma)) \qquad (8)$$

and the transition matrix $\Phi^u\ (t,\sigma)$ is the unique solution of

$$\frac{\partial}{\partial t}\ \Phi^u(t,\ \sigma) = [Dg_0(x(t)) + \sum_{j=1}^{m} u_j(t)Dg_j(x(t))]\Phi^u(t,\sigma) \qquad (9)$$

$$\Phi^u(\sigma,\sigma) \text{ is kxk identity matrix}$$

It is easily seen that $W_v\ (t,\sigma,u)$ exists for all t, $\sigma \geq 0$ and also can be defined in a coordinate free way [5]. Similarly, the input-output map of the adjoint system is given by

$$y^a(t) = \int_0^t W_a(t,\sigma,u)\, u^a(\sigma)\, d\sigma,\ t,\ \sigma \geq 0 \tag{10}$$

and $W_a(t,\sigma,u)$ is determined by $W_v(t,\sigma,u)$ since [5]

$$W_a(t,\sigma,u) = -W_v^T(\sigma,t,u) \text{ for all } u \tag{11}$$

Definition A variational system along an input u is called self-adjoint if

$$W_v(t,\sigma,u) = W_a(t,\sigma,u)(=-W_v^T(\sigma,t,u)) \text{ for all } t,\sigma \geq 0 \tag{12}$$

(in particular $p = m$) □

Note that from (7) it follows that a variational system is self-adjoint if for any input $u^v(t) = u^a(t)$ we have $\frac{d}{dt} p^T(t)\, v(t) = 0$ and hence $p^T(t)\, v(t) = 0$, $t \geq 0$. We can now state the main theorem of this section. Recall that usually a nonlinear system is called minimal if it is observable and accessible. Because an observable and accessible Hamiltonian system is necessarily strongly accessible, we shall henceforth call a system minimal if it is observable and strongly accessible. The following results are contained in [5].

Theorem 2.1 A minimal system $\sum^L$ is Hamiltonian if and only if all the variational systems along any piecewise constant input are self-adjoint. □

We shall only sketch the basic steps in the proof. The "only if" direction is straightforward. For the "if" direction we note that all variational systems are self-adjoint if the input-output maps of the prolongation and of the Hamiltonian extension of $\sum^L$ coincide. If both the prolongation and Hamiltonian extension are minimal, this yields by the Sussmann uniqueness theorem on minimal realizations [22], an isomorphism between the two systems:

Theorem 2.2. A system $\sum^L$ is minimal if and only if the prolongation is minimal, if and only if the Hamiltonian extension is minimal. □

Hence, since $\sum^L$ is minimal, there exists a unique diffeomorphism Φ from the state space TM of the prolongation to the state space T*M of the Hamiltonian extension. The next step is to show that Φ is a vector-bundle isomorphism which is the identity on the base space M, or equivalently, Φ is locally of the form $\Phi(x,v) = (x,\ p = \omega(x)v)$, with $\omega(x)$ an invertible matrix. From the uniqueness of Φ it follows that $\omega(x)$ is skew-symmetric. Hence $\omega: = \sum_{i,\ j=1}^{k} \omega_{ij}(x)\, dx_i \wedge dx_j$, where $\omega_{kj}(x)$ the (i,j)-th element of $\omega(x)$ defines a non-degenerate two-form on M. The hardest part is now to show that ω is also closed, and so defines a symplectic form. It then follows quickly that the system is Hamiltonian with respect to this symplectic form.

We note that the extension of theorem (2.1) to systems $\sum$ is also worked out in [5] and is accomplished by introducing the extended system, $\dot{x} = f(x,v)$, $\dot{v} = u$, $y = h(x,v)$, $u,v \in R^m$, $x \in M$, $y \in R^p$, and applying theorem (2.1).

3. The Input-Output Map Characterization

We state first the initial result, characterizing linear Hamiltonian systems. That is, an input-output map $y(t) = \int_0^t W(t-s)u(s)ds$, $t \geq 0$, which has a finite dimensional linear realization, $\dot{x} = Ax + Bu, y = Cx$, $x(o) = x_o$, has a Hamiltonian realization if and only if $W(t) = -W(-t)^T$ for $t \geq 0$. This result obtained by Brockett and Rahimi [2], bears a very close resemblence to our theorem (2.1); and indeed follows from it since the variational system of a linear system is itself. However it may also be viewed as a distinct result, yielding a characterization of a Hamiltonian system directly in terms of the input-output map. We pursue this for more general systems $\sum$. (Note that for linear systems $W(t)$ is an analytic matrix valued function, so $W(-t)$, $t \geq 0$ is defined from $W(t)$, $t \geq 0$ by analytic continuation).

The linear result was first generalized in Crouch and Irving [3], to input output maps defined by Finite Volterra series. We do not give this result explicitly, but see the remark after theorem (3.3). The next result characterizes secondary Hamiltonian input-output systems in terms of the maps (2). We need some notation in this regard. Let

$$Q(u_1..[u_k,u_{k+1}]\ ...u_n) = Q(u_1...u_ku_{k+1}...u_n) - Q(u_1...u_{k+1}u_k...u_n)$$

and define $Q(u_1...[u_k...u_r]...u_n)$ by induction

$$Q(u_1...[u_k...u_r]...u_n) = Q(u_1...[u_k...u_{r-1}]u_r...u_n) - Q(u_1...u_r[u_k...\ u_{r-1}]...u_n) \tag{13}$$

Theorem 3.1 Jakubczyk [9] [10]

Suppose that an input-output map F has a realization by a minimal input-output system $\sum$ (with Ω compact), then $\sum$ is a secondary Hamiltonian system if and only if either

(i) $Q([u_1...u_k]) = kQ(u_1...u_k)$, $k \geq 2$, $u_i \in \Omega$

or

(ii) $Q([u_1...u_k]v_1...v_r) + Q([v_1....v_r]u_1...u_k) = 0$, k, $r \geq 1$ u_i $v_i \in \Omega$ □ (14)

To determine realizability conditions for Hamiltonian systems $\sum_H$, Jakubczyk introduces the following maps,

$$\tilde{Q}(u_1...u_k) = \frac{\partial}{\partial t_1}\cdots\frac{\partial}{\partial t_{k-1}}\Bigg|_{t_i=0} \int_{u_{k-1}}^{u_k} F((t_1u_1)...(t_{k-1}u_{k-1})u)du \tag{15}$$

Theorem 3.2 Jakubczyk [9] [10]

Suppose an input-output map F has a realization by a minimal input-output system $\sum$ (with Ω compact), then $\sum$ is a Hamiltonian system $\sum_H$ if and only if the maps, $\tilde{Q}$ satisfy either of the conditions (i), or (ii) of theorem (3.1). □

So far the conditions in theorem (3.1) and (3.2) are unrelated to the criteria introduced in the previous section. A link is provided in the following result, see [5],

Theorem 3.3

Consider an input-output map which has a minimal realization by a system $\sum^L$ and which is represented by a Volterra series (3). Then all variational systems of $\sum^L$ are self adjoint if and only if the Volterra kernels satisfy

$$W_n([\sigma_0,\sigma_1\ldots\sigma_k]\sigma_{k+1}\ldots\sigma_n)=(k+1)\ W_n(\sigma_0\ldots\sigma_k\ldots\sigma_n)$$

$$\text{for } k \geq 1,\ n \geq 2. \tag{16}$$

□

Note that theorem (2.1) shows that the conditions (16) are necessary and sufficient for $\sum^L$ to be Hamiltonian. In the case of finite Volterra series this result was proven directly by Crouch and Irving [3]. The bracket operation on the kernels in (16), defined in a similar way as the bracket on Q in (19), was first introduced in Crouch [4]. Again there is no ambiguity in the conditions (16), even though nominally the input-output map provides the kernels. W_n on the domain $\sigma_0 \geq \sigma_1 \geq \ldots \geq \sigma_n$, because for analytic systems the kernels are also analytic, and the values of the kernels on R^{n+1} are determined uniquely by analytic continuation. The relationship between the conditions (16), and the conditions (14) and (15) is not particularly simple because the affine nature of $\sum^L$ is heavily employed to obtain the conditions (16). However, the underlying resemblence between the conditions does have a common link -- that of the Dynkin, Specht, Wever criterion for Lie elements in a free algebra, (See [15]) and comes about because in all three theorems (3.1), (3.2) and (3.3), when $\sum$ is Hamiltonian Q, $\tilde{Q}$ and W_n have the form of iterated Poisson brackets for which the Lie structure is defined by the conditions (14), (15) and (16). This observation then suggests the possibility of generalizing all these results so far obtained to more general Poisson systems, whose underlying geometric structure is a Poisson structure, not a symplectic structure. This generalization is worked out in Jakubczyk [14].

4. The Variational Criterion

In this section we again consider systems $\sum^L$ for simplicity, although similar results are valid for more general systems $\sum$. We shall usually identify a variational field along a given trajectory with the variation of the trajectory itself. We restrict ourselves to piecewise constant inputs and piecewise constant variations, so for example any variational field along an input $\bar{u}$ may be generated by a variation

$$u(t,\varepsilon) = \bar{u}(t) + \varepsilon\ \delta u(t)$$

where $\delta u(t)$ is piecewise constant. The main technical concept is now introduced.

Definition (δu, δy) is called an admissible variation of compact support of $(\bar{u}, \bar{y}) \in \sum_e^+ (0)\ (x_0)$ if

(i) $\delta u(0) = 0$, and supp δu is compact

(ii) supp $\delta y \subset$ supp δu

(iii) Let supp $\delta u \subset (0,T)$ and let $(\bar{u}', \bar{y}') \in \sum_\varepsilon^+ (0)\ (x_0)$ be such that $\bar{u}'(t) = \bar{u}(t)$ and hence $\bar{y}'(t) = \bar{y}(t)$ for $t \in [0,T]$. Define a variation $u'(u,\varepsilon)$ of $\bar{u}'$ by setting $u'(t,\varepsilon) = \bar{u}'(t) + \varepsilon\delta u(t)$. This yields a variation $(u'(t,\varepsilon), y'(t,\varepsilon))$ of $(\bar{u}', \bar{y}')$. We require that the resulting (infinitesimal) variation $(\delta u', \delta y')$ of $(\bar{u}', \bar{y}')$ also satisfies (ii), i.e. supp$\delta y' \subset$ supp $\delta u'$ = supp δu. □

Admissible variations (δu, δy) of $(\bar{u}, \bar{y})$ with supp $\delta u \subset (0,T)$ can be fully characterized in terms of the kernel W_v $(t, \sigma, \bar{u})$ of the variational system (4) along $(\bar{u}, \bar{x}, \bar{y})$ as defined in (8) and (9). Since the transition matrix $\Phi^{\bar{u}}$ in (15) satisfies $\Phi^{\bar{u}}(t,\sigma) = \Phi^{\bar{u}}(t,\sigma)\ \Phi^{\bar{u}}(0,\sigma)$, we may write

$$W_v(t,\sigma,\bar{u}) = G(t,\bar{u})\ H(\sigma,\bar{u})$$

with $G(t,\bar{u})$ an mxk matrix and $H(\sigma,\bar{u})$ a kxm matrix.

Theorem 3.1 Let $\sum^L$ be a minimal system. A variation (δu, δy) of $(\bar{u}, \bar{y}) \in \sum_e^+ (0)\ (x_0)$ with supp $\delta u \subset (0,T)$ is admissible if and only if

$$\int_0^T H(\sigma,\bar{u})\ \delta\ u(\sigma)\ d\sigma = 0 \tag{17}$$

Furthermore let $(\bar{u}, \bar{x}, \bar{y})$ be the corresponding element of $\sum_i^+ (0)\ (x_0)$. Then (δu, δy) is admissible if and only if supp $\delta x \subset$ supp δu, and hence $\delta x\ (T) = 0$. □

Consider now the Hilbert space $H = L_2([0,T], R^m)$, and let D be the dense subspace of piecewise constant right continuous functions. Let S be the finite dimensional subspace of H defined by $S = \{H(\cdot, \bar{u})^T x;\ \in R^k\}$. It follows that δu satisfies (23) if and only if $\delta u \in D \cap S^\perp$, where $\perp$ denotes the orthogonal complement in H. Hence the admissible variations (δu, δy) of $(\bar{u}, \bar{y})$ with supp $\delta u \subset (0,T)$ are in one-to-one correspondence with the functions in $D \cap S^\perp$. Since D is dense in H it follows (c.f. [5]) that $D \cap S^\perp$ is dense in $S^\perp$, and so there are a great many admissible variations of compact support. One of the main results in [5] is

Theorem 3.2 Consider a minimal system $\sum^L$. The system is Hamiltonian if and if for any $(u,y) \in \sum_e^+ (0)\ (x_0)$ and admissible variations $(\delta_i u, \delta_i y)$ of (u,y) with compact support, $i = 1,2$, we have

$$\int_0^\infty [\delta_2^T\ y(t)\ \delta_1\ u(t) - \delta_1^T\ y(t)\ \delta_2 u(t)]\ dt = 0 \tag{18}$$

□

So far we have only considered minimal systems. However the self-adjointness as well as the variational criterion are expressed solely in terms of the (variational) input-output behavior of the system. Also a non-minimal system with external behavior $\sum_e^+ (0)(x_0)$ has a minimal realization with this same external behavior (where minimal means observable and accessible, c.f. [22]). It may therefore be expected that if a non-minimal system satisfies these criteria then a minimal realization will be Hamiltonian. The only flaw in this argument is that an observable and accessible Hamiltonian system is necessarily strongly accessible. Let L be the smallest Lie algebra containing $g_0, g_1, \ldots, g_m$, and L_0 the ideal in L generated by $g_1, \ldots, g_m$.

Theorem 3.4 Let $\sum^L$ be a nonlinear system such that $g_0(x_0) \in L_0(x_0)$. Then a minimal realization of $\sum^L$ is Hamiltonian if and only if every variational system along any piecewise constant control u is self-adjoint, if and only if along any $(u, y) \in \sum_e^+ (0)(x_0)$ and for any two admissible variations $(\delta_i u, \delta_i y)$, i = 1,2, of (u,y) with compact support; condition (18) is satisfied. □

The condition $g_0(x_0) \in L_0(x_0)$, of equivalently $L(x_0) = L_0(x_0)$ simply means that an accessible realization is also strongly accessible. If $g_0(x_0) \notin L_0(x_0)$ then we have to take recourse to time-varying Hamiltonian systems as explained in [8].

Note that for minimal Hamiltonian realizations the internal energy H_0 need not be globally defined. On the other hand there always exists a Hamiltonian realization $\sum_H^L$ for which H_0 is globally and $\sum_H^L$ is quasi-minimal (strongly accessible and weakly observable).

As the complexity of the statement of theorem (3.2) suggests this characterization of Hamiltonian systems is not particularly well suited to the input-output map representation. As we now show it is far better suited to the input-output behavior representation introduced in section 1. We restate our principal definition in this context.

Definition $(\delta u, \delta y)$ is an admissible variation of compact support of $(\bar{u}, \bar{y}) \in \sum_e$ if

(i) supp δu is compact
(ii) supp $\delta y \subset$ supp δu
(iii) Suppose supp $\delta u \subset [T_1, T_2)$, $\bar{x}(T_1) = x_{T_1}$, $\bar{x}(T_2) = x_{T_2}$. Let $(\bar{u}', \bar{x}', \bar{y}') \in \sum_i$ be such that it coincides with $(\bar{u}, \bar{x}, \bar{y})$ for $t \in [T_1, T_2)$. Define a variation $u'(t, \varepsilon)$ of $\bar{u}(t)$ by setting $u'(t,\varepsilon) = \bar{u}(t) + \varepsilon\, \delta u(t)$. This yields a variation $(u'(t, \varepsilon), y'(t, \varepsilon))$ of $(\bar{u}', \bar{y}')$. We require that the resulting (infinitesimal) variation $(\delta u', \delta y')$ of $(\bar{u}', \bar{y}')$ also satisfies (ii), i.e. supp $\delta y' \subset$ supp $\delta u'$ = supp δu. ◘

We obtain a direct analogue of Theorem 3.1 and the following improved version of Theorem 3.2 (See [5].)

Theorem 3.5 Consider a minimal non-initialized system $\sum^L$. Every variational system is self-adjoint (or equivalently (Theorem 2.1), the system is Hamiltonian), if and only if for any $(u, y) \in \sum_e$ all admissible variations $(\delta_i u, \delta_i y)$ of (u, y) with compact support, i = 1,2, satisfy

$$\int_{-\infty}^{+\infty} [\delta_2^T y(t)\delta_1 u(t) - \delta_1^T y(t)\delta_2 u(t)]\, dt = 0 \quad (19)$$

□

Furthermore we have (See [5])

Theorem 3.6 Consider a minimal non-initialized Hamiltonian system $\sum^L$. Let $(u, y) \in \sum_e$ and suppose that (u, y) also belongs to the external behavior $\bar{\sum}_e$ of some other minimal (not necessarily Hamiltonian) system, with the same state space M. Let (Du, Dy) be an admissible variation of (u,y) of compact support where (u, y) is viewed as an element of $\bar{\sum}_e$. If every admissible variation $(\delta u, \delta y)$ of (u, y) with compact support, where (u, y) is viewed as element $\sum_e$, satisfies

$$\int_{-\infty}^{+\infty} [Dy^T(t)\, \delta u(t) - \delta^T y(t)\, Du(t)]\, dt = 0$$

then (Du, Dy) is also an admissible variation with compact support of (u, y) viewed as element of $\sum_e$.

Theorems 3.4 and 3.5 have the following formal interpretation. Consider the "manifold" of maps $N_{M,m}$, defined as the union of all behavior sets $\sum_e$ as $\sum^L$ ranges over all non-initialized minimal systems, with state space M and input space R^m. On this manifold we suppose the "tangent space" to it at (u, y), denoted $T_{(u, y)} N_{M,m}$ as suggested by (19), i.e.,

$$\mu_{(u,y)}((\delta_1 u, \delta_1 y), (\delta_2 u, \delta_2 y)) = \int_{-\infty}^{+\infty} [\delta_2^T y(t)\, \delta_1 u(t) - \delta_1^T; y(t)\, \delta_2 u(t)])\, dt$$

Consider now a Hamiltonian system $\sum$ on M with m inputs. Then $\sum_e$ is a "submanifold" of $N_{M,m}$. Now Theorem 3.4 implies that the sympletic form μ is zero restricted to $\sum_e$, or equivalently, $\sum_e$ is an isotropic submanifold of $N_{M,m}$. On the other hand Theorem 3.5 implies that $\sum_e$ is also a coisotropic submanifold. Hence the following corollary, stated as a conjecture in [20], is formally proven.

Corollary 3.6 A minimal non-initialized system $\sum^L$ on M is Hamiltonian if and only if $\sum_e$ is a Lagrangian submanifold of $N_{M,m}$. □

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